Introduction to the Definite Integral **(Section 5.1)** **Date ____________**
Distance and Accumulated Change

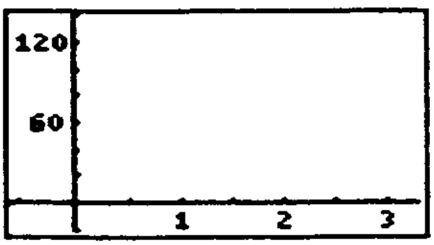
So, what have we studied this far into the course? We began by looking at the "rate of change" of a function. The average rate of change was calculated as the slope of a secant line. Then we estimated an instantaneous rate of change, the slope of the tangent line, by calculating the average rate of change over a small interval. This led to the definition of the derivative function, which tells us the rate of change of a function at any value. Finally, we have looked at applications of the derivative which shows how the derivative can give us more information than just a slope of a tangent line.

Now, we can consider the reverse process. Say we know something about the rate of change of a function. What does this tell us about the function? And why is this important? These are the questions we will address for the remainder of the course.

An Introductory Example

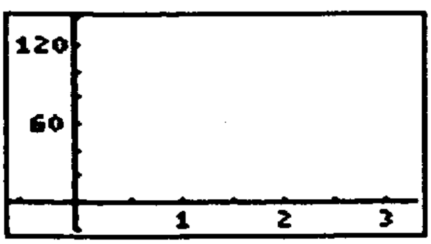
The distance from Tucson to Mesa, Arizona, is approximately 120 miles.

1. If I make this trip at a constant velocity, and it takes me 2 hours, how fast am I traveling? ________________
Sketch a graph of my distance traveled vs. time..



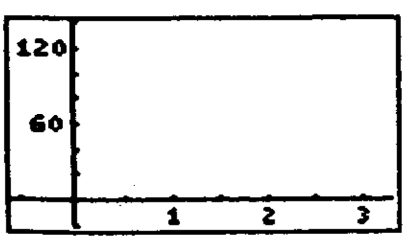
What is the equation of the distance vs. time graph above? ________________

What would the graph of the velocity vs. time graph look like? Sketch it below.



It is easy to see that the slope of the distance vs. time graph is the height of the constant function for the velocity vs. time graph. But, an interesting question is, how could the velocity vs. time graph tell the distance I traveled on the trip?

2. Suppose that instead of driving 60 mph for the entire trip, I drive 40 mph for 3/4 hr. Then I get on a highway and I drive 70 mph for 1 hr, then get off the highway and drive 40 mph again for 1/2 hr. Will I get to my destination (120 miles from Tucson) in that amount of time? Explain!



A More Interesting Example

In the preceding example, the velocity was constant over all of the time intervals. Of course, this is not always the case. We now look at an example where the velocity is continually changing.

Suppose a car is moving and accelerating over a 10 second time interval. And we calculate the car's velocity every two seconds (in ft/sec), obtaining the data shown in the table below.

Time (sec)	0	2	4	6	8	10
Velocity (ft/sec)	5	10	18	31	45	65

The question is: **How far has the car traveled?**

Since we don't know how fast the car is moving at every moment, we can't calculate the distance exactly, but we can make an estimate.

Over the first two seconds the car goes <u>at least</u> ________________ feet. During the next two seconds, it goes at least ________________ feet. So, during the ten-second period the car travels at least:

Distance ≈ ____

This answer is an **underestimate** of the total distance traveled during the ten seconds. How can we obtain an overestimate of the total distance?

Distance ≈ ____

Perhaps a better estimate of the total distance traveled would be to take the average of the underestimate and the overestimate.

Distance ≈ ____________________________

A new, and important, question is: **How can we obtain a more accurate estimate of the total distance traveled?**

The data in the table below is the velocity of the car measured at every second over the ten second period.

Time (sec)	0	1	2	3	4	5	6	7	8	9	10
Velocity (ft/sec)	5	7	10	13	18	24	31	38	45	54	65

Calculate a **lower estimate** of the total distance traveled.

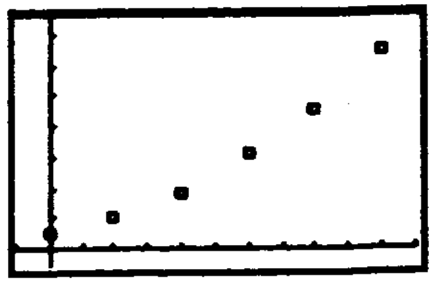
Distance = ____

Calculate an **upper estimate** of the total distance traveled.

Distance = ____

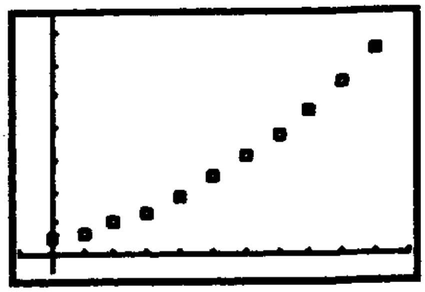
Visualizing the Distance on a Velocity Graph

Consider the two second data in the first table.
The data is recorded graphically in the scatterplot shown. Draw a smooth curve through the data points.



Using the fact that the area of a rectangle is: Area = Base • Height, show the rectangles that represent the under estimate for the total distance traveled by the car. Then show the rectangles that represent the over estimate of the total distance.

Repeat the above procedure to show the rectangles that represent the underestimate and the overestimate for the data recorded every one second.



Notice that as we add more data points, the rectangles used to estimate the distance traveled fit the curve more closely. Fifty rectangles would fit the curve even more closely and give us an even better estimate for the exact distance traveled than using ten rectangles. One hundred rectangles ....

Can you see where we are going?

A Final Example

Land management officials notice that an introduced species of tree is making serious inroads into an ecosystem. In a certain area, the number of new trees per year is increasing every year. Some of the growth rates are shown below.

Year	1950	1960	1970	1980	1990
Trees/Year	337	371	408	448	493

We want to estimate the total number of new trees that have appeared between 1950 and 1990.

1. What is the minimum number of new trees that could have appeared between 1950 and 1990?

2. What is the maximum number of new trees that could have appeared between 1950 and 1990?

3. Suppose that some additional growth figures are obtained, and shown below.

Year	1955	1965	1975	1985
Trees/Year	343	389	418	467

4. Recalculate the under estimate and over estimate using this new information.

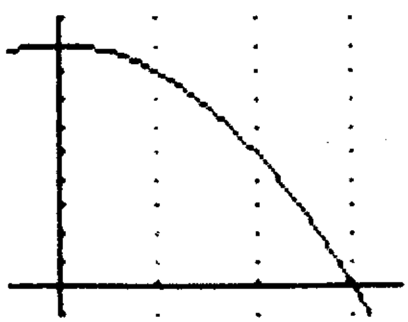
The Definition of the Definite Integral **(Section 5.2)** **Date** ____________

In the previous section we saw how to approximate the total change of a function if we were given the rate of change (represented by a graph or a table). We discovered that this total change was approximated by calculating the areas of rectangles, whose heights were determined by the rate of change function.

Today, we will continue with this discussion of finding areas of rectangles to determine an accumulated sum, but instead of calculating an "under estimate" and an "over estimate", we are going to calculate a "left-hand sum" and a "right-hand sum". Let's look at this procedure with 3 examples; the first with the function represented by a <u>graph</u>, the second example with the function represented by a <u>table of values</u>, and third, the function represented by a <u>function rule</u>.

Example #1: Given the graph of a function f below, approximate the area under the function from $t = 0$ to $t = 3$ with a left-hand sum, and then a right-hand sum. First let the number of rectangles (called n) in your approximation equal 3, then approximate the area with $n = 6$.

n = 3, so $\Delta t =$ ______



Assume each "tick mark" on the graph represents one unit.

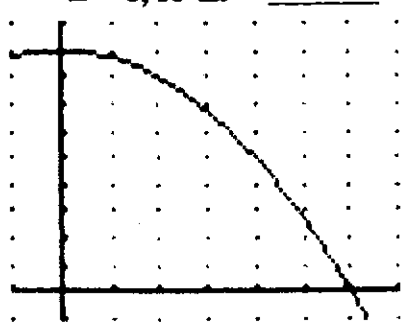
Left-hand sum = ____________________

= ____________________

Right-hand sum = ____________________

= ____________________

n = 6, so $\Delta t =$ ______



The horizontal "tick marks" now represent 0.5 units.

Left-hand sum = ____________________

= ____________________

Right-hand sum = ____________________

= ____________________

Example #2: Given a function represented by the table below, approximate the left-hand sum and the right-hand sum on the interval [6 , 36].

t	6	12	18	24	30	36
f(t)	16	24	30	28	20	22

From the table, n = ________ and $\Delta t =$ ________

Left-hand sum = ______________________________ = ____________

Right-hand sum = ______________________________ = ____________

Example #3: Given the function $f(t) = 2^t$, approximate the area under the function (and above the t-axis) on the interval [-2 , 6] with a left-hand sum and a right-hand sum. Let $n = 4$. Therefore, $\Delta t =$ __________.

Left-hand sum = ______________________________ = ______________________________ = _______

Right-hand sum = ______________________________ = ______________________________ = _______

And, once again, how could we get a better approximation for the exact area under the function?

Left-Hand Sums, Right-Hand Sums, and the Definite Integral

Suppose we have a function $f(t)$ that is continuous for all t in the interval [a , b], and let $f(t) \geq 0$ for all t in [a , b]

We divide the interval from $t = a$ to $t = b$ into n equal subdivisions, each of width Δt, so

$\Delta t =$ ____________________.

Let $t_0, t_1, t_2, \ldots, t_n$ be the endpoints of the subdivisions. (Therefore, $t_0 =$ ______ and $t_n =$ ________.)

Illustrate this by a graph below.

To evaluate the left-hand sum, we use the values of the function from the left endpoint of the interval, and for the right-hand sum, use values of the function from the right end of the interval. Therefore, we have:

Left-hand sum = ____

Right-hand sum = ____

Both the left-hand and right-hand sums can be written more compactly using sigma (or summation) notation.

The symbol Σ is a capital sigma, or Greek letter "S". It is used to represent a "Sum".

It is defined as: $a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i$

Example: Write the series of numbers $1 + 4 + 9 + 16 + 25 + 36 + 49$ using sigma notation.

So, we can write

Right-hand sum $= f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + \ldots + f(t_n)\Delta t =$ ______________________________

In the left hand sum, $f(t_0)\Delta t + f(t_1)\Delta t + f(t_2)\Delta t + \ldots + f(t_{n-1})\Delta t$, we start at i = _____ and stop at i = __________,
so we write

Left-hand sum $= f(t_0)\Delta t + f(t_1)\Delta t + f(t_2)\Delta t + \ldots + f(t_{n-1})\Delta t =$ ______________________________

If f is a rate of change of some quantity, then the left-hand sum and the right-hand sum approximate the total change in the quantity. For most functions, the approximation is improved by ______________________________

To find a better approximation for the **exact total change**, we take larger and larger values of n, and to find the **exact value** we: ______________________________

Definition of the Definite Integral

Suppose f is continuous for $a \le t \le b$. The definite integral of f from a to b, written $\int_a^b f(t)dt$, is the limit of the left-hand or right-hand sums with n subdivisions of [a , b]. (Note: As $n \to$ ________ , $\Delta t \to$ ________).

Therefore, $\int_a^b f(t)dt =$ ______________________ and $\int_a^b f(t)dt =$ ______________________.

Each of these sums is called a ______________________, the function f is called the ______________,

and "a" and "b" are called the ______________________________

So, we have just seen how a definite integral is defined, but this does not tell us how to evaluate a definite integral.

Therefore, a new question is:

How do we calculate definite integrals? And using the function in Example #3, what is $\int_{-2}^{6} 2^t \, dt$?

1. We have already approximated it in Example #3 using left-hand and right-hand sums, and averaging them together.

Left-hand sum = 42.5 Right-hand sum = 170 $\int_{-2}^{6} 2^t \, dt \approx$ ________________

But if we want a good approximation, we need to use a lot of rectangles, and this is a lot of work!

2. We can approximate definite integrals with our calculator. The graphing calculator can compute sums for very large values of n.

This can be done one of two ways with our calculator.

i. From the Home Screen, press the MATH key and select "9:fnInt". (This stands for a "function numerical integral".)

Then to evaluate $\int_{-2}^{6} 2^t \, dt$, enter:

fnInt(______________________________) = ______________

ii. If the function has been graphed on your calculator with the interval included, from the graph select CALC, and 7: $\int f(x)dx$, and enter the limits of integration.

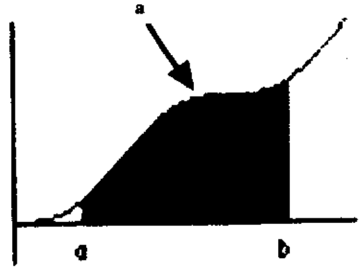
3. For some functions, we are able to calculate the definite integral exactly using geometry. (We will see this in the next class.)

4. The **Fundamental Theorem of Calculus** allows us to evaluate definite integrals exactly by hand, i.e., without our calculator, but you are going to have to wait on this!

The Definite Integral as Area **(Section 5.3)** **Date: ______________**

We have seen that if a function f(x) is continuous (no breaks) and positive (above the x-axis), then $\int_a^b f(x)dx$ represents the area under the graph of f (and above the x-axis) between x = a and x = b.

$$\text{Area} = \int_a^b f(x)dx$$



For some functions, we are able to evaluate this definite integral with geometery.

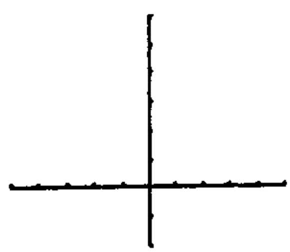
Examples:

For the following problems, sketch the region whose area is given by the definite integral. Use geometry to evaluate the integral, then check your answer with your calculator.

1. $\int_0^2 (-2x+6)dx$

Area = ______________________________

2. $\int_{-3}^3 (4-|x|)dx$



Area = ______________________________

Practice Problems:

3. $\int_2^{12} (8-\frac{1}{2}x)\,dx$

4. $\int_{-3}^3 (|x|+2)\,dx$

The next big question we need to answer is:

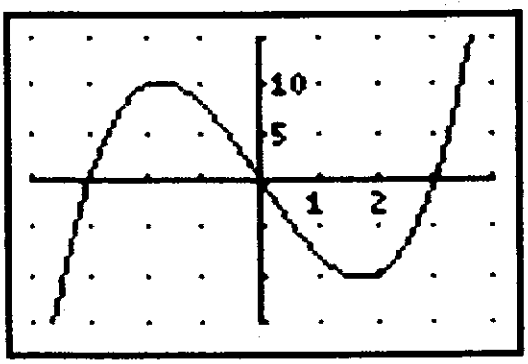
What happens to the definite integral when the function f(x) is not positive on the interval [a , b]?

Perhaps f(x) is negative on the entire interval, or, perhaps, it is positive on part of the interval and negative on another part.

Let's explore this with some examples. We also want to keep in mind the **definition of the definite integral.**

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

Below is the graph of $f(x) = x^3 - 9x$ from x = -4 to x = 4.



Using the graph, estimate $\int_{-3}^{0}(x^3 - 9x)dx$.

$\int_{-3}^{0}(x^3 - 9x)dx \approx$ __________

Evaluate $\int_{-3}^{0}(x^3 - 9x)dx$ with your calculator to see how good the guess was. $\int_{-3}^{0}(x^3 - 9x)dx =$ __________

Use this result to guess, then evaluate, the following definite integrals.

1. $\int_{0}^{3}(x^3 - 9x)dx =$ __________
2. $\int_{-3}^{3}(x^3 - 9x)dx =$ __________
3. $\int_{0}^{-3}(x^3 - 9x)dx =$ __________
4. $\int_{3}^{0}(x^3 - 9x)dx =$ __________
5. $\int_{0}^{2}(x^3 - 9x)dx =$ __________
6. $\int_{-3}^{2}(x^3 - 9x)dx =$ __________
7. What is the total area between the function $f(x) = x^3 - 9x$ and the x-axis from x = -3 to x = 2?

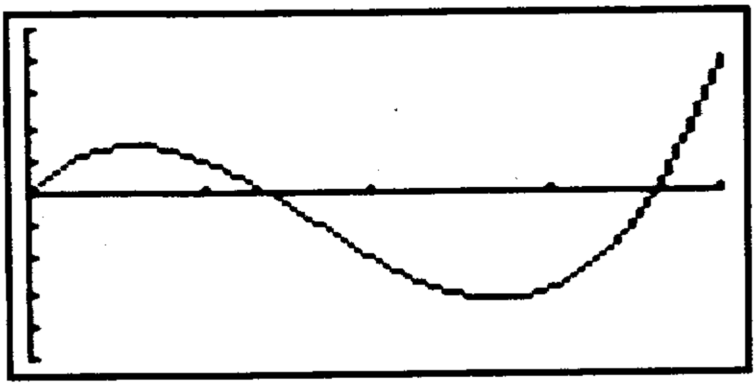
____

In conclusion,

When f(x) is positive for some x-values in the interval [a , b], and negative for other x-values in [a , b], and a < b,

then $\int_a^b f(x)dx$ is ____________________

Example: Interpret the definite integral $\int_0^4 \left(x^3 - 5x^2 + 5x\right)dx$ in terms of area. The figure below shows the graph of $f(x) = x^3 - 5x^2 + 5x$ to help in the analysis.



What is the total area between the x-axis and the function $f(x) = x^3 - 5x^2 + 5x$ from $x = 0$ to $x = 4$?

Area = ____________________

Interpretations of the Definite Integral **(Section 5.4)** **Date:** ________

The objective of this section is to understand the meaning behind the notation $\int_a^b f(x)dx$, and to relate it to other applications, except just finding the "area under the curve".

First, recall the Leibniz notation for the derivative $\frac{dy}{dx}$ to represent $\frac{\Delta y}{\Delta x}$ over a "very small" interval with respect to x. In the same way $\int_a^b f(x)dx$ represents the <u>sum</u> of the areas of a "large number" of rectangles, each with area $f(x_i)\Delta x$. This reminds us that the definite integral is the <u>limit of a sum</u>. The terms that are being added together are products of the form "f(x) times a difference in x".

So what are the <u>units</u> of the result of calculating $\int_a^b f(x)dx$?

They are the product of ____

For example, suppose f(x) represents the rate at which the heart is pumping blood, in liters per second, where x is the time in seconds, express the units and meaning of the integral $\int_0^{10} f(x)dx$.

$\int_0^{10} f(x)dx$ ____

____

Units: ____________________

In general, if f(t) is a function that represents the **rate of change of a quantity**, then

$\int_a^b f(t)dt$ represents ____

Example 1: Assume $f(t) = 60\sqrt{t}$ gives the rate of change of the population of a city, in people per year, at time t years since 1990. If the population of the city is 3000 in 1990, what is the population in 2000?

Example 2: Suppose C(t) represents the cost per day to heat your home in dollars per day, where t is the time measured in days and t = 0 corresponds to January 1, 2009. Interpret the meaning of $\int_0^{90} C(t)dt$ with units.

$\int_0^{90} C(t)dt$ ____

____

Units: __________

The following two problems were taken from **Advanced Placement Calculus Exams** administered to high school calculus students. The first problem contains a trigonometric function, but it is not necessary to understan trigonometry to complete the problem!

Example 3, Question #1a from the 2004 AP Exam.
Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function F defined by

$$F(t) = 82 + 4\sin\left(\frac{t}{2}\right) \quad \text{for } 0 \le t \le 30$$

where F(t) is measured in cars per minute and t is measured in minutes. To the nearest whole number, how many cars pass through the intersection over the 30-minute period?

Example 4, Question #2a, from the 2009 AP Exam.
The rate at which people enter an auditorium for a rock concert is modeled by the function R given by

$$R(t) = 1380t^2 - 675t^3 \quad \text{for } 0 \le t \le 2 \text{ hours}$$

where R(t) is measured in people per hour. One hundred people are in the auditorium at time $t = 0$, when the doo open, and the concert begins at time $t = 2$ when the doors close. Calculate $\int_0^2 R(t)dt$ and explain its meaning, including units.